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Development of an exact dynamic stiffness matrix for free vibration analysis of a twisted Timoshenko beam

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Abstract

An exact dynamic stiffness matrix for a twisted Timoshenko beam is developed in this paper in order to investigate its free vibration characteristics. First the governing differential equations of motion and the associated natural boundary conditions of a twisted Timoshenko beam undergoing free natural vibration are derived using Hamilton's principle. The inclusion of a given pretwist together with the effects of shear deformation and rotatory inertia, gives rise in free vibration to four coupled second order partial differential equations of motion involving bending displacements and bending rotations in two planes. For harmonic oscillation these four partial differential equations are combined into an eighth order ordinary differential equation, which is identically satisfied by all components of bending displacements and bending rotations. This difficult task has become possible only with the help of symbolic computation. Next the exact solution of the differential equation is obtained in completely general form in terms of eight arbitrary constants. This is followed by application of boundary conditions for displacements and forces. The procedure leads to the formation of the dynamics stiffness matrix of the twisted Timoshenko beam relating harmonically varying forces with harmonically varying displacements at its ends. The resulting dynamic stiffness matrix is used with particular reference to the Wittrick–Williams algorithm to compute the natural frequencies and mode shapes of a twisted Timoshenko beam with cantilever end condition. The exact results from the present theory are compared with numerically simulated results using simpler theories, and some conclusions are drawn.
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1. Introduction

There is a considerable amount of published literature on the free vibration analysis of twisted beams [1–17]. Recently, the present author has made a contribution to this literature [18] by investigating the free vibration characteristics of a twisted beam by using the dynamic stiffness

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method for the first time. It is evident from the research reported in the literature that prior to the emergence of the popular finite element method, there were some outstanding works on the subject using exact analytical methods which are based on the solution of classical differential equations of motion. In particular, the works of Troesch et al. [2] and Diprima and Handelman [3] need special mention. These developments are no-doubt significant and they are as relevant today as they were at the time, particularly when solving bench mark problems and validating the finite element and other approximate methods [10,11]. However, a striking shortcoming of these classical analytical works is that they are essentially based on Bernoulli–Euler assumptions and therefore, do not account for the effects of shear deformation and rotatory inertia on the free vibratory motion of the twisted beam.

As the search for refined beam theories has continued over the years the incorporation of the effects of shear deformation and rotatory inertia has progressively become more and more difficult. When developing a refined beam theory the inclusion of these terms is undoubtedly an increased and additional complexity. (Clearly the level of complexity increases as the beam theory becomes more and more refined.) Since Timoshenko implemented the effects of shear deformation and rotatory inertia on a simple Bernoulli–Euler beam in the earlier part of last century [19], research workers have recognized the importance of these effects. These are particularly important for natural frequencies of non-slender beams where the length-to-depth ratios are generally small, and also when higher natural frequencies are required [20]. The term "Timoshenko beam" (which represents a beam theory which accounts for shear deformation and rotatory inertia) is now universally accepted and has featured in literally hundreds of papers, covering amongst others, bending-torsion coupled (metallic) beams [21] and composite beams [22].

It appears that no one has made a serious analytical approach to investigate the free vibration characteristics of a twisted Timoshenko beam of the type analyzed here. It would seem that the expressions needed to derive the governing differential equations for the problem have proved to be too daunting. The problem is further compounded by the fact that even after the governing differential equations have been derived considerable difficulties would unavoidably arise in obtaining an explicit solution in closed analytical form. Clearly, the algebraic expressions for the solution are expected to be extensive and may possibly assume unmanageable proportions. This predominantly algebraic problem can now be overcome because a powerful tool has become available. This is, of course, symbolic computation that has made significant advances in recent years and these have taken place in different directions. It is now considered to be a break through in solid mechanics [23] and there are a number of diverse and wide-ranging symbolic computing packages currently available [24,25].

Against the above background this paper sets out to develop an exact dynamic stiffness matrix for a twisted Timoshenko beam, and then uses it to investigate its free vibration characteristics. It will be shown later that the dynamic stiffness formulation for a twisted Timoshenko beam is considerably more difficult than that of its Bernoulli–Euler counterpart [18]. A secondary object of this paper is to demonstrate a worthwhile application of symbolic computation in solving complex dynamical problems.

The structure analyzed in this paper is a twisted beam, in which the effects of transverse shear in two planes are included. This structure will necessarily undergo torsional deformation as it flexes, but in this paper this induced torsional deformation is ignored. This is a reasonable assumption for beams with doubly symmetric cross-section for which the shear centre and centroid are coincident and as a result the torsional deformation may be assumed to be uncoupled from flexure. However, the present method imposes a serious restriction when analyzing many practical twisted beams such as idealized compressor, turbine and helicopter blades for which the shear centre and centroid are not coincident. In this respect the twisted beam analyzed in this paper exists as an academic concept with limited applications, but has no doubt the merit of introducing structural effects of such complexity that successful analysis is possible only by the use of symbolic computation. Of course, the inclusion of induced twisting response further increases the complexity of the analysis very considerably. For the purpose of demonstrating the make-or-break contribution of the symbolic computation, the simplified twisted beam model that has been chosen—however unreal—is felt justified. The present investigation is considered to be a major step towards developing a more refined theory for twisted beams.

The investigation proceeds with the fundamental assumptions of allowable displacements of a twisted Timoshenko beam in coupled flexural motion. The kinetic and potential energy expressions are derived to formulate the Lagrangian. Hamilton's principle is then applied to derive the governing differential equations in free vibration. The expressions for the shear forces and bending moments at any cross-section of the beam are recovered from the natural boundary conditions, which emerge routinely from the Hamiltonian formulation. By assuming harmonic oscillation the governing partial differential equations are reduced to ordinary differential equations, and are combined into one ordinary differential equation by making substantial use of symbolic computation. The final differential equation is of eighth order and solved in closed analytical form. The boundary conditions for bending displacements, bending rotations, shear forces and bending moments are imposed on the general solution. This enables the elimination of the arbitrary constants from the general solution, leading to the formation of the dynamic stiffness matrix of the twisted Timoshenko beam. Finally using the Wittrick–Williams algorithm [26] the resulting dynamic stiffness matrix is processed to obtain natural frequencies of some chosen examples. The results are discussed and some conclusions are drawn.

2. Theory

A twisted beam of length L is shown in Fig. 1 in a right handed Cartesian co-ordinate system. The global co-ordinate axes XYZ are shown at the left-hand end of the beam whereas the local co-ordinate axes xyz (in lower cases) which vary along the length, as a result of the twist, are shown on the right-hand side. The local y and global Y axes are coincident, both passing through the centroid, and are perpendicular to the beam cross-section, and therefore, represent the axis of

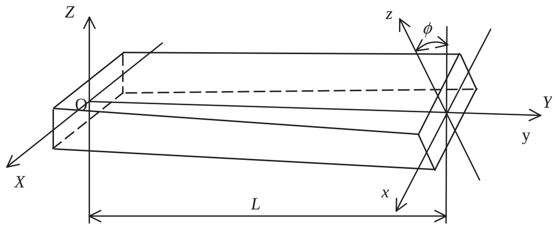

Fig. 1. Axis system and notation used for a twisted Timoshenko beam.

twist of the beam. The rate of twist k is assumed to be constant along the length. Thus, if the twist is zero at the left-hand end, and φ (in radian) at the right-hand end, then $k = \varphi/L$. The area of cross-section of the beam is A and the two principal second moment of areas are I_x and I_z, respectively.

The derivation of the governing (partial) differential equations of motion of a twisted beam (see Fig. 1) undergoing free natural vibration is of some considerable complexity, particularly when the effects of shear deformation and rotatory inertia are taken into account. This is achieved by applying Hamilton's principle (see Appendix A for details). The resulting differential equations in free vibration are presented here for the first time as follows:

$$-\rho A\ddot{u} + k_x AGu'' - k^2 k_z AGu + kAG(k_x + k_z)w' + k_x AG\psi' - kk_z AG\theta = 0, \tag{1}$$

$$-\rho A\ddot{w} + k_z AGw'' - k^2 k_x AGw - kAG(k_x + k_z)u' - kk_x AG\psi - k_z AG\theta' = 0, \tag{2}$$

$$-\rho I_x\ddot{\theta} + EI_x\theta'' - k^2 EI_z\theta - k_z AG\theta - kk_z AGu + k_z AGw' + k(EI_x + EI_z)\psi' = 0, \tag{3}$$

$$-\rho I_x\ddot{\psi} + EI_z\psi'' - k^2 EI_x\psi - k_x AG\psi - kk_x AGw - k_x AGu' - k(EI_x + EI_z)\theta' = 0, \tag{4}$$

where u and w are displacements in the x and z directions of a point lying on the centroidal axis and located at a distance y from the origin, θ and ψ are the corresponding bending rotations about the local x and y axes, ρ, E and G are respectively the density, Young's modulus and modulus of rigidity (shear modulus) of the beam material, ρA is the mass per unit length, EI_x, EI_z, $k_z AG$ and $k_x AG$ are the bending and shear rigidities in the principal planes with k_x and k_z being the shear correction (or shape) factors, and a prime and an over dot represent differentiation with respect to distance y and time t, respectively.

If harmonic variation of u, w, θ, and ψ with circular (angular) frequency ω is assumed then

$$\left.\begin{aligned} u(y,t) &= U(y)\mathrm{e}^{\mathrm{i}\omega t} \\ w(y,t) &= W(y)\mathrm{e}^{\mathrm{i}\omega t} \\ \theta(y,t) &= \Theta(y)\mathrm{e}^{\mathrm{i}\omega t} \\ \psi(y,t) &= \Psi(y)\mathrm{e}^{\mathrm{i}\omega t} \end{aligned}\right\}, \tag{5}$$

where $U(y)$, $W(y)$, $\Theta(y)$ and $\Psi(y)$ are the amplitudes of u, w, θ, and ψ in free vibration.

Substituting Eq. (5) into Eqs. (1)–(4) gives

$$k_x AGU'' - (k^2 k_z AG - \rho A\omega^2)U + kAG(k_x + k_z)W' - kk_z AG\Theta + k_x AG\Psi' = 0, \tag{6}$$

$$-kAG(k_x + k_z)U' + k_z AGW'' - (k^2 k_x AG - \rho A\omega^2)W - k_z AG\Theta' - kk_x AG\Psi = 0, \tag{7}$$

$$-kk_z AGU + k_z AGW' + EI_x\Theta'' - (k^2 EI_z + k_z AG - \rho I_x\omega^2)\Theta + k(EI_x + EI_z)\Psi' = 0, \tag{8}$$

$$-k_x AGU' - kk_x AGW - k(EI_x + EI_z)\Theta' + EI_z\Psi'' - (k^2 EI_x + k_x AG - \rho I_z\omega^2)\Psi = 0. \tag{9}$$

Introducing the non-dimensional variable ξ (in place of y) where

$$\xi = yk = y\frac{\phi}{L}. \tag{10}$$

Eqs. (6)–(9) can be written in the following form:

$$k(D^2 - \mu + b_z^2 s_z^2)U + k(1 + \mu)DW - \mu\Theta + D\Psi = 0, \tag{11}$$

$$k(1+\mu)DU - k(\mu D^2 + b_z^2 s_z^2 - 1)W + \mu D\Theta + \Psi = 0, \tag{12}$$

$$-\mu kU + \mu kDW + (\mu s_x^2 D^2 + b_z^2 r_x^2 s_z^2 - s_z^2 - \mu)\Theta + (\mu s_x^2 + s_z^2)D\Psi = 0, \tag{13}$$

$$kDU + kW + (\mu s_x^2 + s_z^2)D\Theta - (s_z^2 D^2 + b_z^2 r_z^2 s_z^2 - \mu s_x^2 - 1)\Psi = 0, \tag{14}$$

where

$$\mu = \frac{k_z}{k_x}, \quad b_x^2 = \frac{\rho A\omega^2}{EI_x k^4}, \quad b_z^2 = \frac{\rho A\omega^2}{EI_z k^4}, \tag{15}$$

$$s_x^2 = \frac{EI_x k^2}{k_z AG}, \quad s_z^2 = \frac{EI_z k^2}{k_x AG}, \quad r_x^2 = \frac{I_x k^2}{A}, \quad r_z^2 = \frac{I_z k^2}{A} \tag{16}$$

and

$$D = \frac{\mathrm{d}}{\mathrm{d}\xi}. \tag{17}$$

By carrying out an extensive amount of symbolic computation using REDUCE [24], Eqs. (11)–(14) can be combined into one differential equation by eliminating all but one of the variables U, W, Θ or Ψ to obtain, after a lot of simplification, the following eighth order differential equation:

$$(D^8 + C_1 D^6 + C_2 D^4 + C_3 D^2 + C_4)H = 0, \tag{18}$$

where

$$H = U, W, \Theta \quad \text{or } \Psi \tag{19}$$

and

$$C_1 = 4 + b_x^2(r_x^2 + s_x^2) + b_z^2(r_z^2 + s_z^2), \tag{20}$$

$$\begin{aligned} C_2 = {} & (6 - b_x^2 - b_z^2) + b_x^2 r_x^2(1 + b_x^2 s_x^2) + b_z^2 r_z^2(1 + b_z^2 s_z^2) + b_z^2(r_z^2 + s_z^2) \\ & \times \{1 + b_x^2(r_x^2 + s_x^2)\} + b_x^2(r_x^2 + s_x^2) - (b_z^2 r_x^2 + b_x^2 r_z^2), \end{aligned} \tag{21}$$

$$\begin{aligned} C_3 = {} & (4 + 6b_x^2 + 6b_z^2) - (r_x^2 + r_z^2)(b_z^4 s_z^2 + b_x^4 s_x^2) - 2(b_z^2 r_x^2 + b_x^2 r_z^2) \\ & + b_z^2(r_z^2 - s_z^2)(1 + b_x^2 r_x^2) - b_x^2 b_z^2(r_x^2 + s_x^2)(1 - b_z^2 r_z^2 s_z^2) \\ & + b_x^2(r_x^2 - s_x^2)(1 + b_z^2 r_z^2) - b_x^2 b_z^2(r_z^2 + s_z^2)(1 - b_x^2 r_x^2 s_x^2) \\ & - b_x^2 b_z^2\{s_x^2(r_x^2 - s_z^2) + s_z^2(r_z^2 - s_x^2)\}, \end{aligned} \tag{22}$$

$$\begin{aligned} C_4 = {} & (1 - b_x^2)(1 - b_z^2) + b_x^2 b_z^2 r_x^2(1 - b_z^2 r_z^2 s_z^2) + b_z^2(1 - b_x^2 r_x^2 s_x^2)(b_x^2 r_z^2 + b_z^2 s_z^2 - b_z^2 b_x^2 s_z^2 r_z^2) \\ & + b_z^2 s_z^2(b_z^2 r_x^2 + b_x^2 r_z^2) - (1 - b_x^2 s_x^2)(b_z^2 s_z^2 + b_z^2 r_x^2 + b_x^2 r_z^2) - b_x^2 s_x^2(1 - b_x^2) \\ & + b_x^2 b_z^2(r_x^2 r_z^2 - b_x^2 s_x^2 r_x^2 - b_x^2 s_x^2 s_z^2 r_z^2). \end{aligned} \tag{23}$$

The differential Eq. (18) is linear with constant coefficients so that the solution for H (and hence for U, W, Θ and Ψ) can be sought in the form

$$H = \mathrm{e}^{\lambda\xi}. \tag{24}$$

Substituting Eq. (24) into Eq. (18) yields the auxiliary (or characteristic) equation

$$\lambda^8 + C_1\lambda^6 + C_2\lambda^4 + C_3\lambda^2 + C_4 = 0. \tag{25}$$

The eighth order polynomial in λ above, can be reduced to a quartic as follows:

$$p^4 + C_1p^3 + C_2p^2 + C_3p + C_4 = 0, \tag{26}$$

where

$$p = \lambda^2 \quad \text{or} \quad \lambda = \pm\sqrt{p}. \tag{27}$$

The roots of p (and hence of λ) can now be obtained by using standard procedure [27], for example, by factorizing the quartic of Eq. (26) into two quadratics. Note that the roots of p (or λ) can be real or complex depending on the coefficients of the quartic.

Thus the solutions for U, W, Θ and Ψ can be written as

$$U(\xi) = \sum_{j=1}^{8} P_j e^{\lambda_j\xi} = P_1e^{\lambda_1\xi} + P_2e^{\lambda_2\xi} + P_3e^{\lambda_3\xi} + P_4e^{\lambda_4\xi} + P_5e^{\lambda_5\xi} + P_6e^{\lambda_6\xi} + P_7e^{\lambda_7\xi} + P_8e^{\lambda_8\xi}, \tag{28}$$

$$W(\xi) = \sum_{j=1}^{8} Q_j e^{\lambda_j\xi} = Q_1e^{\lambda_1\xi} + Q_2e^{\lambda_2\xi} + Q_3e^{\lambda_3\xi} + Q_4e^{\lambda_4\xi} + Q_5e^{\lambda_5\xi} + Q_6e^{\lambda_6\xi} + Q_7e^{\lambda_7\xi} + Q_8e^{\lambda_8\xi}, \tag{29}$$

$$\Theta(\xi) = \sum_{j=1}^{8} R_j e^{\lambda_j\xi} = R_1e^{\lambda_1\xi} + R_2e^{\lambda_2\xi} + R_3e^{\lambda_3\xi} + R_4e^{\lambda_4\xi} + R_5e^{\lambda_5\xi} + R_6e^{\lambda_6\xi} + R_7e^{\lambda_7\xi} + R_8e^{\lambda_8\xi} \tag{30}$$

and

$$\Psi(\xi) = \sum_{j=1}^{8} S_j e^{\lambda_j\xi} = S_1e^{\lambda_1\xi} + S_2e^{\lambda_2\xi} + S_3e^{\lambda_3\xi} + S_4e^{\lambda_4\xi} + S_5e^{\lambda_5\xi} + S_6e^{\lambda_6\xi} + S_7e^{\lambda_7\xi} + S_8e^{\lambda_8\xi}, \tag{31}$$

where $\lambda_j (j = 1, 2, \ldots, 8)$ are the eight roots of the auxiliary Eq. (25) and P_j, Q_j, R_j, and S_j are four different sets of constants.

It can be shown by substituting Eqs. (28)–(31) into Eqs. (11)–(14) that the constants P_j, Q_j, R_j, and S_j are related as follows:

$$Q_j = \alpha_j P_j, \quad R_j = (\phi\beta_j/L)P_j, \quad S_j = (\phi\gamma_j/L)P_j, \tag{32}$$

where

$$\alpha_j = -\lambda_j A_j/B_j, \quad \beta_j = -A_j/C_j, \quad \gamma_j = -D_j/(\lambda_j C_j) \tag{33}$$

with

$$A_j = b_z^2 s_z^2(-\mu\eta_1 + \mu\varsigma_j + \eta_3) + \mu\rho_j(1 - \eta_1 + \varsigma_j - \eta_3), \tag{34}$$

$$B_j = b_z^2 s_z^2(\mu\eta_1 - \mu\varsigma_j + \eta_3\lambda_j^2) + \mu\rho_j(1 - \eta_1 + \varsigma_j + \eta_3\lambda_j^2), \tag{35}$$

$$C_j = b_z^2 s_z^2(\mu\eta_1 r_x^2 + \eta_2 r_z^2) - \eta_4(\tau_j + \sigma_j + \mu s_x^2 s_z^2 \rho_j^2) - 2\mu\eta_3\kappa_j, \tag{36}$$

$$D_j = b_z^2 s_z^2(\eta_2 + \mu\eta_3\lambda_j^2 - \varepsilon_j) + \mu\rho_j(\mu - \eta_2 + \varepsilon_j + \eta_3\lambda_j^2), \tag{37}$$

$$\eta_1 = 1 - b_z^2 r_z^2 s_z^2, \quad \eta_2 = \mu - b_z^2 r_x^2 s_z^2, \quad \eta_3 = \mu s_x^2 + s_z^2, \quad \eta_4 = \mu + 1, \tag{38}$$

$$\varsigma_j = s_z^2\lambda_j^2 - \mu s_x^2, \quad \rho_j = 1 + \lambda_j^2, \quad \kappa_j = 1 - \lambda_j^2, \tag{39}$$

$$\tau_j = \mu b_z^2 s_x^2 s_z^2(r_z^2\lambda_j^2 - r_x^2), \quad \sigma_j = b_z^2 s_z^4(r_x^2\lambda_j^2 - r_z^2), \quad \varepsilon_j = \mu s_x^2\lambda_j^2 - s_z^2. \tag{40}$$

The expressions for shear force and bending moment in the local axis system (see Appendix A) for harmonic oscillation of the beam, are given by

$$
\begin{aligned}
V_x &= -EI_z k^2[\Psi'' - r\Psi - (r+1)\Theta' + b_z^2 r_z^2 \Psi] \\
&= -EI_z k^2\left[\sum_{j=1}^{8}\lambda_j^2 S_j \mathrm{e}^{\lambda_j\xi} - r\sum_{j=1}^{8} S_j \mathrm{e}^{\lambda_j\xi} - (r+1)\sum_{j=1}^{8}\lambda_j R_j \mathrm{e}^{\lambda_j\xi} + b_z^2 r_z^2 \sum_{j=1}^{8} S_j \mathrm{e}^{\lambda_j\xi}\right] \\
&= -EI_z k^3 \sum_{j=1}^{8}[\gamma_j(\lambda_j^2 - r + b_z^2 r_z^2) - (r+1)\beta_j\lambda_j]P_j \mathrm{e}^{\lambda_j\xi},
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
V_z &= \frac{EI_x}{r}k^2[r\Theta'' - \Theta + (r+1)\Psi' + b_z^2 r_x^2 \Theta] \\
&= \frac{EI_x}{r}k^2\left[r\sum_{j=1}^{8}\lambda_j^2 R_j \mathrm{e}^{\lambda_j\xi} - \sum_{j=1}^{8} R_j \mathrm{e}^{\lambda_j\xi} + (r+1)\sum_{j=1}^{8}\lambda_j S_j \mathrm{e}^{\lambda_j\xi} + b_z^2 r_x^2 \sum_{j=1}^{8} R_j \mathrm{e}^{\lambda_j\xi}\right] \\
&= \frac{EI_x}{r}k^3 \sum_{j=1}^{8}[\beta_j(r\lambda_j^2 - 1 + b_z^2 r_x^2) + (r+1)\gamma_j\lambda_j]P_j \mathrm{e}^{\lambda_j\xi}
\end{aligned}
\tag{42}
$$

and

$$
M_x = -EI_x k(\Theta' + \Psi) = -EI_x k\left[\sum_{j=1}^{8}\lambda_j R_j \mathrm{e}^{\lambda_j\xi} + \sum_{j=1}^{8} S_j \mathrm{e}^{\lambda_j\xi}\right] = -EI_x k^2 \sum_{j=1}^{8}(\beta_j\lambda_j + \gamma_j)P_j \mathrm{e}^{\lambda_j\xi}, \tag{43}
$$

$$
M_z = -EI_z k(\Psi' - \Theta) = -EI_z k\left[\sum_{j=1}^{8}\lambda_j S_j \mathrm{e}^{\lambda_j\xi} - \sum_{j=1}^{8} R_j \mathrm{e}^{\lambda_j}\right] = -EI_z k^2 \sum_{j=1}^{8}(\gamma_j\lambda_j - \beta_j)P_j \mathrm{e}^{\lambda_j\xi}, \tag{44}
$$

where a prime now denotes differentiation with respect to ξ and

$$
r = \frac{EI_x}{EI_z}. \tag{45}
$$

The dynamic stiffness matrix of the twisted beam can now be obtained by applying the boundary condition for displacements and forces at its ends.

The boundary conditions for the bending displacement and bending rotation are:

At the left-hand end:

$$
y = 0\,(\xi = 0){:}\ U = U_{x1}, \quad W = W_{z1}, \quad \Theta = \Theta_{x_1}, \quad \Psi = \Psi_{z_1}. \tag{46}
$$

At the right-hand end:

$$
y = L\,(\xi = \phi = kL){:}\ U = U_{x2}, \quad W = W_{z2}, \quad \Theta = \Theta_{x_2}, \quad \Psi = \Psi_{z2}. \tag{47}
$$

The boundary conditions for the shear force and bending moment are:

At the left-hand end:

$$
y = 0\,(\xi = 0){:}\ V_x = V_{x1}, V_z = V_{z1}, M_x = M_{x1}, M_z = M_{z1} \tag{48}
$$

At the right-hand end:

$$
y = 0\,(\xi = \phi = kL){:}\ V_x = -V_{x2}, V_z = -V_{z2}, M_x = -M_{x2}, M_z = -M_{z2} \tag{49}
$$

Substituting Eqs. (46) and (47) into Eqs. (28)–(31) gives

$$U_{x1} = U(0) = \sum_{j=1}^{8} P_j, \tag{50}$$

$$W_{z1} = W(0) = \sum_{j=1}^{8} \alpha_j P_j, \tag{51}$$

$$\Theta_{x1} = \Theta(0) = k \sum_{j=1}^{8} \beta_j P_j, \tag{52}$$

$$\Psi_{z1} = \Psi(0) = k \sum_{j=1}^{8} \gamma_j P_j \tag{53}$$

and

$$U_{x2} = U(\phi) = \sum_{j=1}^{8} P_j \mathrm{e}^{\lambda_j \phi}, \tag{54}$$

$$W_{z2} = W(\phi) = \sum_{j=1}^{8} \alpha_j P_j \mathrm{e}^{\lambda_j \phi}, \tag{55}$$

$$\Theta_{x2} = \Theta(\phi) = k \sum_{j=1}^{8} \beta_j P_j \mathrm{e}^{\lambda_j \phi}, \tag{56}$$

$$\Psi_{z2} = \Psi(\phi) = k \sum_{j=1}^{8} \gamma_j P_j \mathrm{e}^{\lambda_j \phi}, \tag{57}$$

Eqs. (50) and (57) can be written in matrix form as follows:

$$\begin{bmatrix} U_{x1} \\ W_{z1} \\ \Theta_{x1} \\ \Psi_{z1} \\ U_{x2} \\ W_{z2} \\ \Theta_{x2} \\ \Psi_{z2} \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} & R_{17} & R_{18} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} & R_{27} & R_{28} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} & R_{37} & R_{38} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} & R_{47} & R_{48} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} & R_{57} & R_{58} \\ R_{61} & R_{62} & R_{63} & R_{64} & R_{65} & R_{66} & R_{67} & R_{68} \\ R_{71} & R_{72} & R_{73} & R_{74} & R_{75} & R_{76} & R_{77} & R_{78} \\ R_{81} & R_{82} & R_{83} & R_{84} & R_{85} & R_{86} & R_{87} & R_{88} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \end{bmatrix} \tag{58}$$

or

$$\boldsymbol{\delta} = \mathbf{RP}, \tag{59}$$

where the elements of $\mathbf{R}$ (for $j = 1, 2, 3 \ldots 8$) are given by

$$R_{1j} = 1, \tag{60}$$

$$R_{2j} = \alpha_j, \tag{61}$$

$$R_{3j} = k\beta_j, \tag{62}$$

$$R_{4j} = k\gamma_j, \tag{63}$$

$$R_{5j} = e^{\lambda_j \phi}, \tag{64}$$

$$R_{6j} = \alpha_j e^{\lambda_j \phi}, \tag{65}$$

$$R_{7j} = k\beta_j e^{\lambda_j \phi}, \tag{66}$$

$$R_{8j} = k\gamma_j e^{\lambda_j \phi}. \tag{67}$$

Substituting Eqs. (48)–(49) into Eqs. (41)–(44), gives

$$V_{x1} = V_x(0) = -EI_z k^3 \sum_{j=1}^{8} [\gamma_j(\lambda_j^2 + b_z^2 r_z^2 - r) - (r+1)\beta_j \lambda_j] P_j, \tag{68}$$

$$V_{z1} = V_z(0) = \frac{EI_x}{r} k^3 \sum_{j=1}^{8} [\beta_j(r\lambda_j^2 + b_z^2 r_x^2 - 1) + (r+1)\gamma_j \lambda_j] P_j, \tag{69}$$

$$M_{x1} = M_x(0) = -EI_x k^2 \sum_{j=1}^{8} (\beta_j \lambda_j + \gamma_j) P_j, \tag{70}$$

$$M_{z1} = M_z(0) = -EI_z k^2 \sum_{j=1}^{8} (\gamma_j \lambda_j - \beta_j) P_j \tag{71}$$

and

$$V_{x2} = V_x(\phi) = EI_z k^3 \sum_{j=1}^{8} [\gamma_j(\lambda_j^2 + b_z^2 r_z^2 - r) - (r+1)\beta_j \lambda_j] P_j e^{\lambda_j \phi}, \tag{72}$$

$$V_{z2} = V_z(\phi) = -\frac{EI_x}{r} k^3 \sum_{j=1}^{8} [\beta_j(r\lambda_j^2 + b_z^2 r_x^2 - 1) + (r+1)\gamma_j \lambda_j] P_j e^{\lambda_j \phi}, \tag{73}$$

$$M_{x2} = M_x(\phi) = EI_x k^2 \sum_{j=1}^{8} (\beta_j \lambda_j + \gamma_j) P_j e^{\lambda_j \phi}, \tag{74}$$

$$M_{z2} = M_z(\phi) = EI_z k^2 \sum_{j=1}^{8} (\gamma_j \lambda_j - \beta_j) P_j e^{\lambda_j \phi}, \tag{75}$$

Eqs. (68)–(75) can be written in matrix form as follows:

$$\begin{bmatrix} V_{x_1} \\ V_{z_1} \\ M_{x_1} \\ M_{z_1} \\ V_{x2} \\ V_{z2} \\ M_{x2} \\ M_{z2} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{16} & Q_{17} & Q_{18} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} & Q_{25} & Q_{26} & Q_{27} & Q_{28} \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} & Q_{35} & Q_{36} & Q_{37} & Q_{38} \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} & Q_{45} & Q_{46} & Q_{47} & Q_{48} \\ Q_{51} & Q_{52} & Q_{53} & Q_{54} & Q_{55} & Q_{56} & Q_{57} & Q_{58} \\ Q_{61} & Q_{62} & Q_{63} & Q_{64} & Q_{65} & Q_{66} & Q_{67} & Q_{68} \\ Q_{71} & Q_{72} & Q_{73} & Q_{74} & Q_{75} & Q_{76} & Q_{77} & Q_{78} \\ Q_{81} & Q_{82} & Q_{83} & Q_{84} & Q_{85} & Q_{86} & Q_{87} & Q_{88} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \end{bmatrix} \tag{76}$$

or

$$\mathbf{F} = \mathbf{QP}, \tag{77}$$

where the elements of $\mathbf{Q}$ (for $j = 1, 2, \ldots, 8$) are given by

$$Q_{1j} = -EI_z k^3[\gamma_j(\lambda_j^2 + b_z^2 r_z^2 - r) - (r+1)\beta_j\lambda_j], \tag{78}$$

$$Q_{2j} = \frac{EI_x}{r} k^3[\beta_j(r\lambda_j^2 + b_z^2 r_x^2 - 1) + (r+1)\gamma_j\lambda_j], \tag{79}$$

$$Q_{3j} = -EI_x k^2(\beta_j\lambda_j + \gamma_j), \tag{80}$$

$$Q_{4j} = -EI_z k^2(\gamma_j\lambda_j - \beta_j) \tag{81}$$

and

$$Q_{5j} = EI_z k^3[\gamma_j(\lambda_j^2 + b_z^2 r_z^2 - r) - (r+1)\beta_j\lambda_j]\mathrm{e}^{\lambda_j\phi}, \tag{82}$$

$$Q_{6j} = -\frac{EI_x}{r} k^3[\beta_j(r\lambda_j^2 + b_z^2 r_x^2 - 1) + (r+1)\gamma_j\lambda_j]\mathrm{e}^{\lambda_j\phi}, \tag{83}$$

$$Q_{7j} = EI_x k^2(\beta_j\lambda_j + \gamma_j)\mathrm{e}^{\lambda_j\phi}, \tag{84}$$

$$Q_{8j} = EI_z k^2(\gamma_j\lambda_j - \beta_j)\mathrm{e}^{\lambda_j\phi}. \tag{85}$$

The constant vector $\mathbf{P}$ can now be eliminated from Eqs. (59) and (77) to give

$$\mathbf{F} = \mathbf{QR}^{-1}\boldsymbol{\delta} = \mathbf{K}\boldsymbol{\delta}, \tag{86}$$

where

$$\mathbf{K} = \mathbf{QR}^{-1} \tag{87}$$

is the required dynamic stiffness matrix.

When computing the dynamic stiffness matrix $\mathbf{K}$, it should be noted that the roots of p and hence for λ, see Eqs. (25) and (26), can be complex and as a consequence, the elements of matrices $\mathbf{Q}$ and $\mathbf{R}$ can be complex. Therefore, the matrix inversion and multiplication steps of Eq. (87) must be carried out using complex arithmetic. The resulting dynamic stiffness matrix $\mathbf{K}$ will, of course, be symmetric and real, with imaginary parts of each element being zero.

Thus, the force displacement relationship at the nodes of the harmonically vibrating twisted Timoshenko beam is given by

$$\begin{bmatrix} V_{x1} \\ V_{z_1} \\ M_{x_1} \\ M_{z_1} \\ V_{x2} \\ V_{z2} \\ M_{x2} \\ M_{z2} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\ & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} \\ & & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} \\ & & & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} \\ & & & & K_{55} & K_{56} & K_{57} & K_{58} \\ & S & Y & M & & K_{66} & K_{67} & K_{68} \\ & & & & & & K_{77} & K_{78} \\ & & & & & & & K_{88} \end{bmatrix} \begin{bmatrix} U_{x1} \\ W_{z1} \\ \Theta_{x_1} \\ \Psi_{z1} \\ U_{x2} \\ W_{z2} \\ \Theta_{x_2} \\ \Psi_{z_2} \end{bmatrix}. \tag{88}$$

It is now necessary to transform the above relationship to global co-ordinates using an appropriate transformation. Clearly, the displacements and forces at the left-hand end of the twisted beam are already in global co-ordinates, whereas the corresponding displacements and forces at the right-hand end are in local co-ordinates (see Figs. 1 and 2).

Referring to Fig. 2, the shear forces and bending moments at the right hand of the element can be resolved from global to local co-ordinates as follows. (Note that the lower and upper cases have been respectively used for the suffices of shear forces and bending moments to indicate whether they correspond to local or global co-ordinates.)

$$V_{x_2} = V_{X_2}\cos\phi - V_{Z_2}\sin\phi, \tag{89}$$

$$V_{z_2} = V_{X_2}\sin\phi + V_{Z_2}\cos\phi, \tag{90}$$

$$M_{x_2} = M_{X_2}\cos\phi - M_{Z_2}\sin\phi, \tag{91}$$

$$M_{z_2} = M_{X_2}\sin\phi + M_{Z_2}\cos\phi. \tag{92}$$

Thus, the relationships for the shear force and bending moment between the global and local co-ordinates at both ends of the beam element can be expressed as

$$\begin{bmatrix} V_{x_1} \\ V_{z_1} \\ M_{x_1} \\ M_{z1} \\ V_{x_2} \\ V_{z_2} \\ M_{x_2} \\ M_{z_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & -s & 0 & 0 \\ 0 & 0 & 0 & 0 & s & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & 0 & 0 & 0 & s & c \end{bmatrix} \begin{bmatrix} V_{X_1} \\ V_{Z_1} \\ M_{X_1} \\ M_{Z_1} \\ V_{X_2} \\ V_{Z_2} \\ M_{X_2} \\ M_{Z_2} \end{bmatrix}, \tag{93}$$

where

$$c = \cos\phi \tag{94}$$

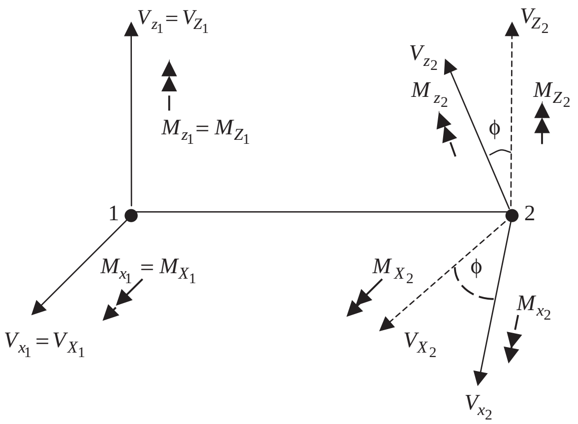

Fig. 2. Shear forces and bending moments at the ends of the twisted Timoshenko beam shown in local and global co-ordinates.

and

$$s = \sin \phi. \tag{95}$$

The displacements can be transformed from the global to local co-ordinates exactly in the same way as the forces by making use of the above transformation matrix **T**, given by

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & -s & 0 & 0 \\ 0 & 0 & 0 & 0 & s & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & 0 & 0 & 0 & s & c \end{bmatrix}. \tag{96}$$

In this way the stiffness matrix of the twisted beam in global co-ordinates $\tilde{\mathbf{K}}$ can now be formulated as

$$\tilde{\mathbf{K}} = \mathbf{T}'\mathbf{K}\mathbf{T}, \tag{97}$$

where $\mathbf{T}'$ denotes the transpose of the transformation matrix **T**.

3. Application of Wittrick–Williams algorithm

The dynamic stiffness matrix of Eq. (88) can now be used to compute the natural frequencies and mode shapes of twisted Timoshenko beams with various end conditions. A non-uniform twisted Timoshenko beam can also be analyzed for its free vibration characteristics by idealizing it as an assemblage of many uniform twisted Timoshenko beams. An accurate and reliable method of calculating the natural frequencies and mode shapes of a structure using the dynamic stiffness method is to apply the well-known algorithm of Wittrick and Williams [26] which has featured in numerous papers [28,29]. Before applying the algorithm the dynamic stiffness matrices of all individual elements in a structure are to be assembled to form the overall dynamic stiffness matrix $\mathbf{K}_f$ of the final (complete) structure, which may, of course, consist of a single element. The algorithm monitors the Sturm sequence condition of $\mathbf{K}_f$ in such a way that there is no possibility of missing a frequency (or mode) of the structure. This is, of course, not possible in the conventional finite element method. The algorithm (unlike its proof) is very simple to use and because of its extensive coverage in the literature the procedure is not repeated here, but interested investigators are recommended to read Refs. [28,29].

4. Scope of the theory and restrictions

The twisted beam considered in this paper is assumed to behave according to the Timoshenko theory which accounts for the effects of shear deformation and rotatory inertia, but the beam has a constant rate of twist along its length and is assumed to exhibit coupling between bending displacements only. These displacements are considered to be uncoupled with torsional and/or extensional deformations. Also the cross-section of the beam is not allowed to warp. These assumptions are quite legitimate for many twisted beams with doubly symmetric cross-section, but they can be severe for many other (practical) twisted beams such as helicopter or turbine blades for which coupling between bending, torsional and extensional deformations and the rotational speed can have significant effects. The dynamic stiffness development of such complex twisted beams involves much more difficulty requiring additional insights.

5. Results and discussion

The dynamic stiffness theory developed above for a twisted Timoshenko beam is applied to a cantilever blade taken from the literature [13,18]. This example is particularly suitable for investigating the degenerate case when the effects of shear deformation and rotatory inertia are ignored so that the results become directly comparable with published results. However, in order to validate the theory and demonstrate the accuracy of results using the Timoshenko theory, the well-established computer program BUNVIS-RG [30,31] that evaluates the dynamic stiffness method for untwisted Timoshenko beam elements is used. When modelling a twisted beam and preparing data for BUNVIS-RG a large number of untwisted beam elements with an appropriate orientation of each was utilized. The example blade [13,18] has a length $L = 3.048\,\mathrm{m}$ with a substantial angle of twist which is zero at the root and 40° at the tip so that the rate of twist $k = \phi/L = 0.22905$ radian/m. The structural and other properties used are: (i) $E = 70 \times 10^9\,\mathrm{N/m^2}$, (ii) $G = 27 \times 10^9\,\mathrm{N/m^2}$, (iii) $\rho = 2700\,\mathrm{kg/m^3}$, (iv) $A = 0.0127667\,\mathrm{m^2}$, (vi) $EI_x = 2869.7\,\mathrm{N\,m^2}$, (vii) $EI_z = 57393\,\mathrm{N\,m^2}$, (vii) $k_x = 2/3$, and (viii) $k_z = 5/8$.

The first five natural frequencies of the blade obtained from the present theory are shown in Table 1 together with those obtained from the Bernoulli–Euler theory [18]. Shear deformation and rotatory inertia have had very little effect on the natural frequencies of this particular blade. This is because the blade is extremely slender and the key parameters r_x, r_z, s_x and s_z which principally affect the natural frequencies when using the Timoshenko theory are very small for the problem. These are $r_x = 0.00041$, $r_z = 0.00184$, $s_x = 0.00081$, and $s_z = 0.00362$. The slenderness ratios of the

Table 1
Natural frequencies of a twisted blade [13,18] with cantilever end condition

Frequency number	Natural frequency (rad/s)	
	Bernoulli–Euler theory [18]	Present theory
1	3.4717	3.4715
2	13.347	13.340
3	25.171	25.165
4	56.372	56.363
5	103.26	103.20

blade in the two principal planes ($(L/\sqrt{I_x/A})$ and $(L/\sqrt{I_z/A})$) being, respectively, around 1700 and 380 are thus very large. Of course, the shear deformation and rotatory inertia are not expected to have any major effect on the natural frequencies of such a very slender blade. However, the results of Table 1 indicate that when the effects of shear deformation and rotatory inertia are taken into account, the natural frequencies decrease as expected.

A more appropriate comparison is made with the same illustrative example as above, but with the bending rigidities EI_x and EI_z increased by a factor of 5000. This gives the Timoshenko beam parameters $r_x = 0.029$, $r_z = 0.130$, $s_x = 0.057$, and $s_z = 0.256$. The first three natural frequencies of the twisted cantilever beam obtained using the present theory are shown in Table 2 alongside the results obtained using the Bernoulli–Euler theory [18]. The significant difference in results between the Timoshenko and the Bernoulli–Euler theories is apparent, particularly for the second and third natural frequencies of the twisted beam. Similar results were obtained using BUNVIS-RG [30,31] with a large number of untwisted beam elements. It has earlier been established [18,32] that approximate results of a discretized structure using a number of uniform straight elements converge almost parabolically to the exact result with increasing number of elements. The first three natural frequencies for the cantilever obtained from BUNVIS-RG [30,31] using 10 and 20 elements and their parabolic limits are shown in Table 3 using both the Bernoulli–Euler and the Timoshenko theories. A comparison of results illustrated in Tables 2 and 3 shows good agreement, and they clearly indicate that the parabolic limit of the approximate natural frequencies agrees to five-figure accuracy to the exact ones which is in accord with earlier investigations [18,32].

The final set of results using the present theory was obtained to demonstrate the effects of slenderness ratio on the natural frequencies of a twisted Timoshenko beam. Using a cantilever twisted Timoshenko beam with the same rate of twist as above, a comparison of natural frequencies was made between this beam and that of an untwisted beam. For both twisted and untwisted beams the computed natural frequencies with the inclusion of the effects of shear deformation and rotatory inertia (ω_n^T) are compared with those (ω_n^B) obtained when using the Bernoulli–Euler theory [18]. The percentage difference in the result $((\omega_n^B - \omega_n^T)/\omega_n^B) \times 100\%$ is plotted against the slenderness ratio (L/r_G) of the twisted beam, where the radius of gyration r_G is defined as $r_G = \sqrt[4]{(I_x I_z/A^2)}$. The plot for the first three natural frequencies ($n = 1$, 2 and 3) for the twisted beam is shown in Fig. 3 by solid lines whereas the corresponding results for the untwisted beam are shown by broken lines. These results demonstrate that the effects of shear deformation and rotatory inertia on the natural frequencies of a twisted beam are similar to those encountered in an untwisted beam. However, the effects are seen to be marginally more pronounced in the case

Table 2
Effects of shear deformation and rotatory inertia on the natural frequencies of a cantilever twisted beam

Frequency number	Natural frequency (rad/s)		
	Bernoulli–Euler theory [18] $r_x = r_z = s_x = s_z = 0$	Present theory $r_x = 0.029$, $r_z = 0.130$, $s_x = 0.057$, $s_z = 0.256$	% difference
1	245.50	239.09	2.61
2	943.38	769.77	18.40
3	1779.6	1425.5	19.90

Table 3
Natural frequencies of a twisted beam computed by BUNVIS-RG [30,31] using straight untwisted beam elements

Frequency number	Natural frequency (rad/s)					
	Bernoulli–Euler theory			Timoshenko theory		
	10 elements	20 elements	Parabolic limit	10 elements	20 elements	Parabolic limit
1	245.47	245.49	245.50	239.06	239.08	239.09
2	946.26	944.10	943.38	771.01	770.08	769.77
3	1769.7	1777.1	1779.6	1421.9	1424.6	1425.5

of a twisted beam than that of an untwisted one. As expected the effects are more significant for higher natural frequencies and smaller slenderness ratios.

6. Conclusions

The governing differential equations of motion of a twisted Timoshenko beam undergoing free natural vibration are derived applying Hamilton's principle. These are subsequently used to develop the dynamic stiffness matrix of the twisted Timoshenko beam. The application of the dynamic stiffness matrix is demonstrated by numerical results with particular reference to the Wittrick–Williams algorithm. The exact results from the present theory are validated by predicting the parabolic limit of approximate results obtained from the idealization of a twisted Timoshenko beam using a large number of untwisted beam elements. The shear deformation and rotatory inertia have similar effects on the natural frequencies of a twisted beam as they have on an untwisted beam. The research presented in this paper can be used as an aid to validate the finite element and other approximate methods, and is expected to stimulate further research on the dynamic stiffness development of complex structural elements.

Appendix A. Derivation of the governing differential equations of motion of a twisted Timoshenko beam

Hamiltonian mechanics is used to derive the governing differential equations of motion of a freely vibrating twisted Timoshenko beam, which has a uniform rate of twist along its length.

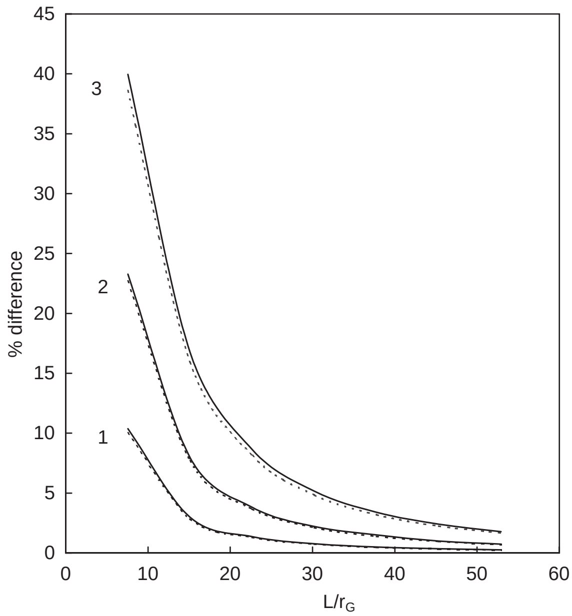

Fig. 3. The effects of shear deformation and rotatory inertia on the variation of the first three natural frequencies of a twisted and an untwisted beams as a function of slenderness ratio. —— twisted beam ($\phi = 40°$, $k = 0.22905\,\text{rad/m}$), ----- untwisted beam ($\phi = 0°$, $k = 0\,\text{rad/m}$).

A set of allowable displacements and rotations is used as the starting point to form the system of direct and shearing strains. The expressions for strain energy and kinetic energy are then derived and subsequently used when applying Hamilton's principle.

In Fig. 4, $O(X, Y, Z)$ is an inertial frame, with OY along the line of centroids of the undeflected beam cross-sections. Let G be the centroid at $Y = y$, and Gx and Gz principal axes in bending of the cross-section. The two-dimensional axis system in the plane of the cross-section represented by $G(x, z)$ has a right handed rotation ϕ about OY, so that the angle between Gx and OX (and also between Gz and OZ) is ϕ as shown. This is the angle of twist at y so that the rate of twist k (which is assumed to be constant) is $\partial\phi/\partial y$.

Let the local displacements be u along Gx and w along Gz, and θ and ψ be the rotations of the cross-section about the x- and z-axis, respectively. Now consider an adjacent section at $Y = y + \mathrm{d}y$, and let $\bar{G}(\bar{x}, \bar{z})$ be the corresponding axis system, see Fig. 5. Allowing for the relative rotation $\mathrm{d}\phi = k\,\mathrm{d}y$ of the element $\mathrm{d}y$, the axial and shearing strains of a point P on the Gxz-plane due to bending and shearing actions can be derived by identifying the elastic distortion of an element of sides $\mathrm{d}y$, $\mathrm{d}x$ and $\mathrm{d}z$ as follows.

A.1. Expression for shearing strains and strain energy due to shear

Since the displacements at G are u along Gx and w along Gz, the corresponding displacements at $\bar{G}$ are $u + (\partial u/\partial y)\,\mathrm{d}y$ along $\bar{G}\bar{x}$ and $w + (\partial w/\partial y)\,\mathrm{d}y$ along $\bar{G}\bar{z}$. Now to examine the shearing strains γ_{xy} and γ_{zy} at P (see Fig. 5) it is necessary to determine the change in the right angle formed

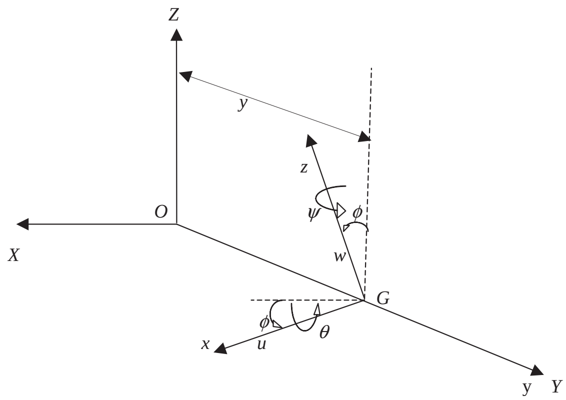

Fig. 4. Displacements and rotations of the centroid G at a distance y from the origin of the twisted Timoshenko beam shown in local co-ordinates.

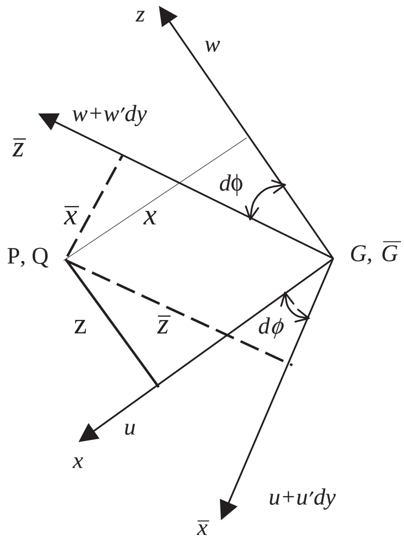

Fig. 5. Displacements of the centroids G and $\bar{G}$ at the left and right-hand ends of an elemental length dy of the twisted Timoshenko beam shown in local co-ordinates.

by lines drawn through P parallel to Gx and Gy for the evaluation of γ_{xy} and lines drawn parallel to Gz and Gy for the evaluation of γ_{zy}.

The displacements $u + (\partial u/\partial y)\,\mathrm{d}y$ along $\bar{G}\bar{x}$ and $w + (\partial w/\partial y)\,\mathrm{d}y$ along $\bar{G}\bar{z}$ will have the following components (see Fig. 5):

$$\left(u + \frac{\partial u}{\partial y}\mathrm{d}y\right)\cos(\mathrm{d}\phi) + \left(w + \frac{\partial w}{\partial y}\mathrm{d}y\right)\sin(\mathrm{d}\phi) \quad \text{along } Gx$$

and

$$-\left(u + \frac{\partial u}{\partial y}\mathrm{d}y\right)\sin(\mathrm{d}\phi) + \left(w + \frac{\partial w}{\partial y}\mathrm{d}y\right)\cos(\mathrm{d}\phi) \quad \text{along } Gz.$$

Using the general rules for infinitesimal calculus and neglecting higher order terms so that $\cos(\mathrm{d}\varphi) \cong 1$ and $\sin(\mathrm{d}\varphi) \cong \mathrm{d}\varphi$ in the limit, the above components can now be written as

$$u + \frac{\partial u}{\partial y}\mathrm{d}y + w\frac{\partial \phi}{\partial y}\mathrm{d}y, \quad \text{along } Gx$$

and

$$-u\frac{\partial \phi}{\partial y}\mathrm{d}y + w + \frac{\partial w}{\partial y}\mathrm{d}y, \quad \text{along } Gz.$$

Thus, the relative displacements in planes $\bar{G}\bar{x}\bar{z}$ and Gxz are

$$\left(\frac{\partial u}{\partial y} + w\frac{\partial \phi}{\partial y}\right)\mathrm{d}y = \left(\frac{\partial u}{\partial y} + kw\right)\mathrm{d}y, \quad \text{along } Gx$$

and

$$\left(-u\frac{\partial \phi}{\partial y} + \frac{\partial w}{\partial y}\right)\mathrm{d}y = \left(-ku + \frac{\partial w}{\partial y}\right)\mathrm{d}y, \quad \text{along } Gz,$$

where $k = (\partial\phi/\partial y)$ is the rate of twist.

Therefore, the contributions to the shearing strains γ_{xy} and γ_{zy} as a result of the above displacements are respectively, given by

$$\left(\frac{\partial u}{\partial y} + kw\right) \quad \text{and} \quad \left(-ku + \frac{\partial w}{\partial y}\right).$$

The effect of the right-handed rotation θ about Gx (or Px) and ψ about Gz (or Pz) is to modify the shearing strains, and we have, defining the shear strain as the reduction in a right angle in a defined plane, the final expressions for the shearing strains γ_{xy} and γ_{zy} as follows (see Fig. 6)

$$\gamma_{xy} = \psi + \frac{\partial u}{\partial y} + kw \tag{A.1}$$

and

$$\gamma_{zy} = -\theta + \frac{\partial w}{\partial y} - ku. \tag{A.2}$$

From these identified shearing strains, the expression for the strain energy $\mathcal{U}_S$ due to transverse shear can now be written as

$$\begin{aligned}\mathcal{U}_S &= \frac{1}{2}\int\int_V\int G(\gamma_{xy}^2 + \gamma_{yz}^2)\,\mathrm{d}v = \int_0^L \int_A G(k_x\gamma_{xy}^2 + k_z\gamma_{yz}^2)\,\mathrm{d}A\,\mathrm{d}y \\ &= \frac{1}{2}\int_0^L \{k_xGA(\gamma_{xy})^2 + k_zGA(\gamma_{yz})^2\}\,\mathrm{d}y \\ &= \frac{1}{2}\int_0^L \{k_xGA(\psi + u' + kw)^2 + k_zGA(-\theta + w' - ku)^2\}\,\mathrm{d}y,\end{aligned} \tag{A.3}$$

where a prime denotes differentiation with respect to y.

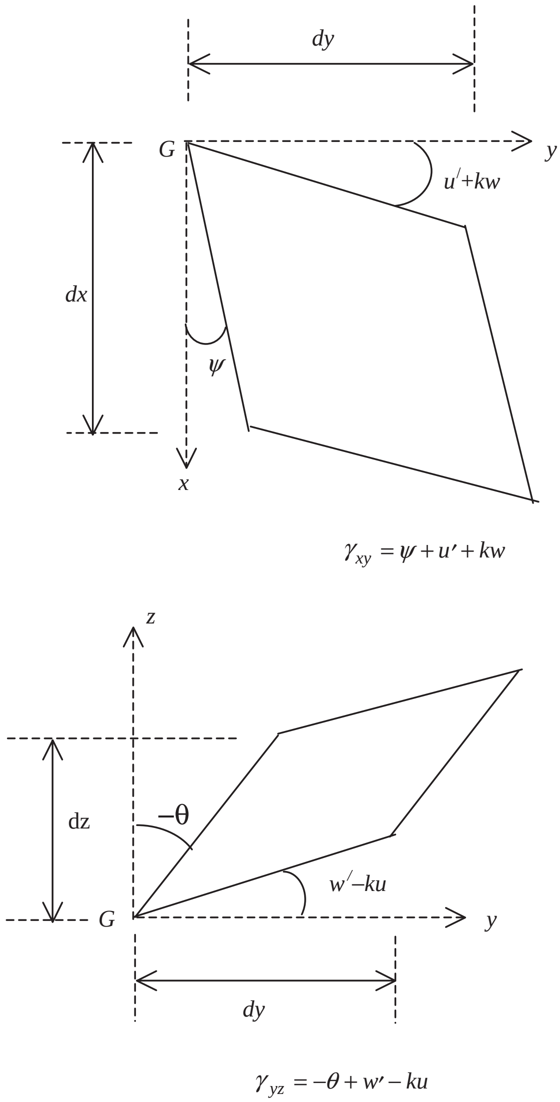

Fig. 6. Shearing strains of an infinitesimal element dy in the xy and yz planes.

A.2. Expression for normal strain and strain energy due to normal strain

In order to determine the axial strain ε_y at P in the Gxz plane as a result of bending actions, consider a line through P, parallel to OY, which meets the section $\bar{G}\bar{x}\bar{z}$ at Q, say (see Fig. 5), then PQ for the undeformed beam is of length dy. Now if one finds the relative displacements of Q to P in the Y direction, the normal strain at P follows at once, as follows.

The co-ordinates of P in the Gxz and $\bar{G}\bar{x}\bar{z}$ axis systems are, respectively, (x, z) and $(x - kz\,\mathrm{d}y, kx\,\mathrm{d}y + z)$, see Fig. 5. The bending rotations of the plane Gxz about the axes Gx and Gz being θ and ψ, the corresponding bending rotations of the plane $\bar{G}\bar{x}\bar{z}$ about $\bar{G}\bar{x}$ and $\bar{G}\bar{z}$ are $\theta + \theta'\,\mathrm{d}y$ and $\psi + \psi'\,\mathrm{d}y$, respectively.

In section Gxz the displacement of P in the Y direction is $-z\theta + x\psi$ whereas in section $\bar{G}\bar{x}\bar{z}$ the displacement of Q in the Y direction is $-\bar{z}(\theta + \theta' \mathrm{d}y) + \bar{x}(\psi + \psi' \mathrm{d}y)$.

Thus the axial displacement of Q is

$$
\begin{aligned}
&-(z + kx\,\mathrm{d}y)(\theta + \theta'\mathrm{d}y) + (x - kz\,\mathrm{d}y)(\psi + \psi'\mathrm{d}y) \\
&\quad = -z\theta + x\psi + (-kx\theta - z\theta' - kz\psi + x\psi')\,\mathrm{d}y.
\end{aligned}
$$

Noting that the axial displacement of P is $-z\theta + x\psi$, the normal strain ε_y at P can now be written as

$$
\varepsilon_y = -kx\theta - z\theta' - kz\psi + x\psi' = x(\psi' - k\theta) - z(\theta' + k\psi). \tag{A.4}
$$

Therefore, the strain energy $\mathcal{U}_B$ due to axial strain resulting from bending actions is given by

$$
\mathcal{U}_B = \frac{1}{2}\int\int_V\int E\varepsilon_y^2\,\mathrm{d}v = \frac{1}{2}\int_0^L \int_A E\varepsilon_y^2\,\mathrm{d}A\,\mathrm{d}y = \frac{1}{2}\int_0^L \left[E\int_A \{x(\psi' - k\theta) - z(\theta' + k\psi)\}^2 \mathrm{d}A\right]\mathrm{d}y. \tag{A.5}
$$

Noting that Gx and Gz are principal axes so that $\int\int_A xz\,\mathrm{d}A = 0$, the above expression simplifies to

$$
\mathcal{U}_B = \frac{1}{2}\int_0^L \{EI_x(\theta' + k\psi)^2 + EI_z(\psi' - k\theta)^2\}\,\mathrm{d}y, \tag{A.6}
$$

where I_x and I_z are the second moment of areas of the cross-section about the principal axes Gx and Gz, respectively, and defined in the usual notation as follows:

$$
I_x = \int\int_A z^2\,\mathrm{d}A, \quad I_z = \int\int_A x^2\,\mathrm{d}A. \tag{A.7}
$$

A.3. Total strain energy due to bending and shearing actions

The total strain energy $\mathcal{U}$ can now be obtained by adding the strain energy due to shearing strain resulting from the shear load and strain energy due to normal strain resulting from bending actions, given by Eqs. (A.3) and (A.6), respectively. Thus,

$$
\begin{aligned}
\mathcal{U} = \mathcal{U}_B + \mathcal{U}_S = \frac{1}{2}\int_0^L &[\{EI_x(\theta' + k\psi)^2 + EI_z(\psi' - k\theta)^2\} \\
&+ \{k_x AG(\psi + u' + kw)^2 + k_z AG(-\theta + w' - ku)^2\}]\,\mathrm{d}y
\end{aligned} \tag{A.8}
$$

A.4. Expression for the kinetic energy

The kinetic energy of the twisted Timoshenko beam can be formulated from the velocity components of the point P on the Gxz-plane as follows. These velocity components can be obtained by taking the time derivative of the displacements of P in the Gx, Gy and Gz directions. Clearly the velocity of the point P along Gx, Gy and Gz are $\dot{u}$, $-z\dot{\theta} + x\dot{\psi}$, and $\dot{w}$, respectively, with an over dot representing the time derivative.

So the kinetic energy $\mathscr{J}$ of the twisted beam can be formulated as

$$\mathscr{J} = \frac{1}{2}\int\int_V\int \rho\{\dot{u}^2 + (-z\dot{\theta} + x\dot{\psi})^2 + (\dot{w})^2\}\,\mathrm{d}v = \frac{1}{2}\int_0^L \{\rho A(\dot{u} + \dot{w}) + \rho I_x\dot{\theta}^2 + \rho I_z\dot{\psi}^2\}\,\mathrm{d}y, \quad \text{(A.9)}$$

where ρ is the density of the beam material.

A.5. Application of Hamilton's principle and derivation of the governing differential equations

Using the expressions for potential and kinetic energies $\mathscr{U}$ and $\mathscr{J}$ of the twisted beam (see Eqs. (A.8) and (A.9)) Hamilton's principle can now be applied to derive the governing differential equations of motion in free vibration.

Hamilton's principle states that

$$\delta\int_{t_1}^{t_2} \mathscr{L}\,\mathrm{d}t = 0, \quad \text{(A.10)}$$

where $\mathscr{L} = \mathscr{J} - \mathscr{U}$ is the Lagrangian, and the integration is taken between arbitrary intervals of time t_1 and t_2 for a dynamic trajectory.

Substituting the expressions for $\mathscr{U}$ and $\mathscr{J}$ from Eqs. (A.8) and (A.9) for the Lagrangian $\mathscr{L}$ in Eq. (A.10) and using the δ operator, integrating by parts and noting that δu, δw, $\delta\theta$ and $\delta\psi$ are completely arbitrary, the governing differential equations of motion of the twisted Timoshenko beam are derived. As a by product of the Hamiltonian formulation the associated natural boundary conditions which give the expressions for shear forces and bending moments are also obtained. The whole procedure for the present problem is tedious, but merely a mathematical process. The derivation is routinely carried out by successive integrations, the details of which are too extensive to report. The final expressions are given as follows.

A.6. Governing differential equations of motion in free vibration

$$-\rho A\ddot{u} + k_xAGu'' - k^2k_zAGu + kAG(k_x + k_z)w' + k_xAG\psi' - kk_zAG\theta = 0, \quad \text{(A.11)}$$

$$-\rho A\ddot{w} + k_zAGw'' - k^2k_xAGw - kAG(k_x + k_z)u' - kk_xAG\psi - k_zAG\theta' = 0, \quad \text{(A.12)}$$

$$-\rho I_x\ddot{\theta} + EI_x\theta'' - k^2EI_z\theta - k_zAG\theta - kk_zAGu + k_zAGw' + k(EI_x + EI_z)\psi' = 0, \quad \text{(A.13)}$$

$$-\rho I_z\ddot{\psi} + EI_z\psi'' - k^2EI_x\psi - k_xAG\psi - kk_xAGw - k_xAGu' - k(EI_x + EI_z)\theta' = 0, \quad \text{(A.14)}$$

A.7. Natural boundary conditions

Shear force in the x direction:

$$V_x = -k_xAGu' - k_xAG\psi - kk_xAGw. \quad \text{(A.15)}$$

Shear force in the z direction:

$$V_z = -k_zAGw' + k_zAG\theta + kk_zAGu. \quad \text{(A.16)}$$

Bending moment about the x-axis:

$$M_x = -EI_x\theta' - kEI_x\psi. \tag{A.17}$$

Bending moment about the z-axis:

$$M_z = -EI_z\psi' + kEI_z\theta. \tag{A.18}$$

With the help of the governing differential Eqs. (A.13) and (A.14), the expressions for the shear forces V_x and V_z in Eqs. (A.15) and (A.16) can be written in the following alternative forms:

$$V_x = -EI_z\psi'' + k^2EI_x\psi + k(EI_x + EI_z)\theta' + \rho I_z\ddot{\psi}, \tag{A.19}$$

$$V_z = EI_x\theta'' - k^2EI_z\theta + k(EI_x + EI_z)\psi' - \rho I_x\ddot{\theta}. \tag{A.20}$$

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